Symmetries and a hierarchy of the general KdV equation



1987 J. Phys. A: Math. Gen. 20 359

(http://iopscience.iop.org/0305-4470/20/2/021)

J. Phys. A: Math. Gen. **20** (1987) 359-366. Printed in the UK

# Symmetries and a hierarchy of the general KdV equation

Tian Chou†
International Centre for Theoretical Physics, Miramare, Trieste, Italy

Received 26 February 1986

**Abstract.** Two groups (old and new) of symmetries and their Lie algebra properties for the KdV and cylindrical KdV equations are unified and extended to the general KdV equation.

## 1. Introduction

As is well known, there are two groups (old and new) of symmetries for the KdV equation (Ibragimov and Shabat 1979, Chen *et al* 1982) and two groups of symmetries for the cylindrical KdV equation (Chen and Zhu 1984, Olver 1980). In Chen and Zhu (1984), the authors pointed out, without proof, that these symmetries satisfied a Lie algebra. Recently, Li and Zhu (1985) gave a proof for the KdV equation and obtained some more results. In this paper, we will unify and extend these results for the KdV and cylindrical KdV equations to the general KdV equation.

This paper is organised as follows. We introduce some notation and some well known results in § 2, and then in §§ 3 and 4 we find the strong symmetry (recursion operator) and two groups of symmetries for the general KdV equation by using different methods. Finally, we prove that these symmetries satisfy a Lie algebra.

## 2. Notations and lemmas

Let $U$ be a set of functions such that $u \in U$ and the derivatives of $u$ of any order with respect to $x$ and $t$ tend to zero rapidly as $|x| \to \infty$. In what follows, we always assume that $u \in U$.

Let $G(x, t, u) = G(x, t, u, u_x, \ldots)$. $G$ can be a function or an operator, and $G'(u)[r]$ (or simply, $G'[r]$) is called the derivative of $G$ in the direction $r$:

$$G'(u)[r] = (\partial/\partial\varepsilon) G(u + \varepsilon r)|_{\varepsilon=0} \qquad r \in U. \tag{2.1}$$

This yields

$$(\phi K)'[r] = \phi'[r]K + \phi K'[r] \tag{2.2}$$

where $\phi$ is an operator and $K$ is a function.

We consider the evolution equation

$$u_t = K(x, t, u, u_x, \ldots) \tag{2.3}$$

which depends on $x$ and $t$ explicitly.

† Permanent address: Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, People's Republic of China.

0305-4470/87/020359+08$02.50  

*Definition.* $\sigma(x, t, u)$ is called a symmetry of (2.3) if $\sigma$ satisfies the linear equation

$$\mathrm{d}\sigma/\mathrm{d}t = K'[\sigma]$$

where $\mathrm{d}\sigma/\mathrm{d}t$ is the total derivative and $u$ satisfies equation (2.3).

Sometimes we call the vector field

$$X = \sigma(u)(\partial/\partial u) \tag{2.4}$$

or the flow generated by $X$

$$u = u(u_0, \varepsilon) \tag{2.5}$$

a symmetry of (2.3) (Fuchssteiner and Oevel 1982, Stramp 1984). Flow $u(u_0, \varepsilon)$ is the solution of the equation

$$\mathrm{d}u/\mathrm{d}\varepsilon = \sigma(u)$$

and satisfies the initial condition $u(u_0, 0) = u_0$. Hence, the flow (2.5) (or $X$) leaves equation (2.3) invariant. In fact, this invariant property can also be used as the definition and we can extend the definition to the general differential equation

$$F(x, t, v, u, v_x, u_x, v_t, u_t, \ldots) = 0 \tag{2.6}$$

i.e. $(\theta, \sigma)$ is called a symmetry of (2.6) if $(\theta, \sigma)$ satisfies

$$F'[\theta, \sigma] = 0$$

where $\theta$ and $\sigma$ correspond to $v$ and $u$ respectively (Fuchssteiner and Oevel 1982, Stramp 1984).

*Definition.* The operator $\Phi$ is called a strong symmetry or a recursion operator of (2.3) if it maps the symmetry to the symmetry of (2.3), i.e. if $\sigma$ is a symmetry of (2.3), then $\Phi\sigma$ is also a symmetry of (2.3).

*Definition.* The operator $\Phi$ is called a hereditary symmetry if

$$\Phi'[\Phi a]b - \Phi'[\Phi b]a = \Phi(\Phi'[a]b - \Phi'[b]a)$$

is valid for any functions $a$ and $b$.

It is not difficult to prove the following lemmas (Oevel and Fokas 1984, Fuchssteiner 1981).

*Lemma 1.* $\sigma$ is a symmetry of (2.3) if and only if

$$\sigma'[K] - K'[\sigma] + \partial\sigma/\partial t = 0$$

where $\partial\sigma/\partial t$ is the partial derivative of $\sigma$ to $t$.

*Lemma 2.* If the operator $\Phi$ satisfies

$$\mathrm{d}\Phi/\mathrm{d}t = [K', \Phi]$$

($[K', \Phi] = K' \circ \Phi - \Phi \circ K'$), i.e.

$$\Phi'[K] + \partial\Phi/\partial t = [K', \Phi]$$

then $\Phi$ is a strong symmetry of (2.3).

*Lemma 3.* $\Phi$ is a hereditary symmetry if and only if

$$\Phi^2[a, b]+[\Phi a, \Phi b]=\Phi([\Phi a, b]+[a, \Phi b])$$

for any functions $a$ and $b$.

## 3. Strong symmetry of the general KdV equation

We consider the general KdV (GKdV) equation

$$u_t+u_{xxx}+6uu_x+6f(t)u-x(f'+12f^2)=0 \tag{3.1}$$

where $f(t)$ is an arbitrary function of $t$. When $f=0$, it is the KdV equation

$$u_t+u_{xxx}+6uu_x=0$$

and when $f(t)=1/12t$, it is the cylindrical KdV equation

$$u_t+u_{xxx}+6uu_x+u/2t=0.$$

When $f=C_0$ ($C_0$ is an arbitrary constant), (3.1) is reduced to

$$u_t+u_{xxx}+6uu_x+6C_0u-12C_0x^2=0.$$

In Tian Chou (1985), we have found the Lax pair for the GKdV equation (3.1):

$$\Omega=\begin{pmatrix} 0 & 1 \\ k-u & 0 \end{pmatrix} \mathrm{d}x+\begin{pmatrix} u_x+2f & -(4k+2u) \\ u_{xx}-(k-u)(4k+2u) & -(u_x+2f) \end{pmatrix} \mathrm{d}t \tag{3.2}$$

where $k=xf(t)+\lambda g(t)$, $\lambda$ is an arbitrary constant, and $g=\exp(-\int 12f\,\mathrm{d}t)(g'+12gf=0)$. Hence, (3.1) can be considered as the completely integrable condition of the following linear equations:

$$\begin{aligned} v_{xx}&=(k-u)v \\ v_t&=-(4k+2u)v_x+(u-4k)_xv. \end{aligned} \tag{3.3}$$

To obtain the strong symmetry, we look for the symmetry of equation (3.3) first, i.e. the solution $\theta$ and $\sigma$ of the equations

$$\begin{aligned} \theta_x&=(k-u)\theta-\sigma v \\ \theta_t&=-(4k+2u)\theta_x-2\sigma v_x+\sigma_xv+(u-4k)_x\theta. \end{aligned} \tag{3.4}$$

It can be shown that

$$\theta=v\mathrm{D}^{-1}v^2 \tag{3.5}$$

$$\sigma=-2(v^2)_x=-4vv_x \tag{3.6}$$

are the symmetries of (3.3), where $\mathrm{D}=\mathrm{d}/\mathrm{d}x$, $\mathrm{D}^{-1}$ is the inverse operator of D.

Furthermore, we look for the flow generated by

$$X=\theta\,\partial/\partial v+\sigma\,\partial/\partial u \tag{3.7}$$

i.e. the $v(v_0, \varepsilon)$ and $u(v_0, u_0, \varepsilon)$ which satisfy the equations

$$\mathrm{d}v/\mathrm{d}\varepsilon=v\mathrm{D}^{-1}v^2 \qquad \mathrm{d}u/\mathrm{d}\varepsilon=-2(v^2)_x \tag{3.8}$$

and the initial conditions $v(v_0, 0) = v_0$, $u(v_0, u_0, 0) = u_0$. In a similar way to Stramp (1984), we have

$$v = v_0(1 + \varepsilon D^{-1} v_0^2) \tag{3.9}$$

$$u = u_0 + 2[\ln(1 + \varepsilon D^{-1} v_0^2)]_{xx}. \tag{3.10}$$

(3.10) can be considered as a Bäcklund transformation of the GKdV equation and the extension of the Bäcklund transformation of the KdV equation (Stramp 1984, Weiss *et al* 1983).

Substituting $\sigma = -2(v^2)_x = -4vv_x (v^2 = -\frac{1}{2}D^{-1}\sigma)$ into

$$v_{xx} = (k - u)v$$

we obtain

$$D\sigma + 4v_x^2 + 2uD^{-1}\sigma = 2kD^{-1}\sigma'.$$

Differentiating the last equation with respect to $x$ and substituting $k = xf + \lambda g$, $v_{xx} = (k - u)v$ into it, we obtain

$$D^2\sigma + 4u\sigma + 2(u_x - f)D^{-1}\sigma = 4(xf + \lambda g)\sigma$$

i.e.

$$(1/g(t))[D^2 + 4(u - xf) + 2(u_x - f)D^{-1}]\sigma = 4\lambda\sigma. \tag{3.11}$$

Suppose

$$\Phi = (1/g)[D^2 + 4(u - xf) + 2(u_x - f)D^{-1}].$$

Since

$$\Phi\sigma = 4\lambda\sigma \qquad d\sigma/dt = K'[\sigma]$$

is a completely integrable system, the completely integrable condition

$$d\Phi/dt = [K', \Phi]$$

is established. Therefore $\Phi$ is a strong symmetry of (3.1). In particular, when $f = 0$, $g = 1$ and $\Phi = D^2 + 4u + 2u_x D^{-1}$, this is a well known strong symmetry of the KdV equation, when $f = 1/12t$, $g = 1/12t$ and $\Phi = 12t[D^2 + 4(u - x/12t) + 2(u_x - 1/12t)D^{-1}]$, this is a strong symmetry of the cylindrical KdV equation (Fuchssteiner 1981).

It is not difficult to check that $\Phi$ is a hereditary symmetry as well. Therefore, a hierarchy of the GKdV equation is generated by $\Phi$ and (3.1):

$$u_t = K_m \qquad K_m = \Phi^m K \qquad m = 0, 1, 2, \ldots \tag{3.12}$$

and $\Phi$ is the strong symmetry of all of equations (3.12).

We point out that there is a transformation which links the KdV equation $v_\tau + v_{\xi\xi\xi} + 6vv_\xi = 0$ to GKdV equation (3.1) (Calogero 1985):

$$u = gv + xf \qquad \xi = xg^{1/2} \qquad \tau = \int g^{3/2}\, dt. \tag{3.13}$$

But this transformation could not transform KdV equations of high order to the general KdV equations of high order. We can derive the strong symmetry, symmetries and the Lie algebra relations of the GKdV equation by using (3.13). In this paper, we use a different method.

## 4. Two groups of symmetries of GKdV equations

Since zero can be considered as a trivial symmetry of the GKdV equation (3.1), naturally we consider

$$\sigma = (1/g)(h(t)u_x + m(t))$$

or

$$\sigma = (1/g)[h(t)(u_x - f) + l(t)] \tag{4.1}$$

as a symmetry of (3.1). Substituting (4.1) into

$$\mathrm{d}\sigma/\mathrm{d}t = K'[\sigma]$$

i.e.

$$\mathrm{d}\sigma/\mathrm{d}t = -(\sigma_{xxx} + 6u\sigma_x + 6u_x\sigma + 6f\sigma)$$

we have the following conditions on $h$ and $l$:

$$h' + 6fh + 6l = 0$$

$$l' + 18fl - 6f^2h - h'f = 0.$$

Then we have

$$l = C_0 g^2$$

and

$$h = g^{1/2}\left(C_1 + 6C_0 \int g^{3/2}\,\mathrm{d}t\right)$$

($C_0$ and $C_1$ are arbitrary constants). Therefore

$$\sigma = g^{-1/2}\left(C_1 + 6C_0 \int g^{3/2}\,\mathrm{d}t\right)(u_x - f) + C_0 g.$$

If we take $C_0 = 0$, $C_1 = 1$, we obtain

$$\sigma_0 = (1/\sqrt{g})(u_x - f).$$

If we take $C_0 = \frac{1}{2}$, $C_1 = 0$, we obtain

$$\tau_0 = 3g^{-1/2} \int g^{3/2}\,\mathrm{d}t(u_x - f) + \tfrac{1}{2}g.$$

Therefore, two groups (old and new) of symmetries are generated by $\sigma_0$, $\tau_0$ and $\Phi$:

$$\sigma_n = \Phi^n \sigma_0 \qquad n = 0, 1, 2, \ldots$$

$$\tau_n = \Phi^n \tau_0 \qquad n = 0, 1, 2, \ldots.$$

In particular, for the KdV equation, we have $\sigma_n = K_n$ $(n = 0, 1, 2, \ldots)$.

## 5. Lie algebra of symmetries of the GKdV equation

*Theorem.* $\sigma_n$ and $\tau_n$ $(n = 0, 1, 2, \ldots)$ satisfy a Lie algebra

$$[\sigma_m, \sigma_n] = 0 \tag{5.1}$$

$$[\sigma_m, \tau_n] = (2m+1)\sigma_{m+n-1} \qquad m + n \geqslant 1 \tag{5.2}$$

$$[\tau_m, \tau_n] = 2(m-n)\tau_{m+n-1} \qquad m + n \geqslant 1 \tag{5.3}$$

$([a, b] = a'[b] - b'[a]$, $m, n = 0, 1, 2, \ldots)$.

To prove this theorem, we need the following lemmas.

*Lemma 4.* $\Phi'[\sigma_m]=[\sigma'_m,\Phi]=\sigma'_m\circ\Phi-\Phi\circ\sigma'_m$.

*Proof.* Since

$$\begin{aligned}\Phi'[\sigma_0]&=4\sigma_0+2(\sigma_0)_x D^{-1}\\&=g^{-3/2}[4(u_x-f)+2u_{xx}D^{-1}]\\\Phi\circ\sigma'_0&=\Phi\circ(1/\sqrt{g})D\\&=g^{-3/2}[D^3+4(u-xf)D+2(u_x-f)]\\\sigma'_0\circ\Phi&=(1/\sqrt{g})D\circ\Phi\\&=g^{-3/2}[D^3+4(u-xf)D+6(u_x-f)+2u_{xx}D^{-1}]\end{aligned}$$

then

$$\Phi'[\sigma_0]+\Phi\circ\sigma'_0-\sigma'_0\circ\Phi=0. \tag{5.4}$$

It is not difficult to check that (5.5) is equivalent to

$$\Phi[\sigma_0,a]=[\sigma_0,\Phi a]$$

for any function $a$ and we say that $\Phi$ commutes with $\sigma_0$. Since $\Phi$ is a hereditary symmetry, according to lemma 3, we can prove that $\Phi$ commutes with $\sigma_n$ $(n=1,2,\ldots)$ as well. Therefore

$$\Phi'[\sigma_m]+\Phi\circ\sigma'_m-\sigma'_m\circ\Phi=0\qquad m=0,1,2,\ldots.$$

*Lemma 5.*

$$\begin{aligned}[\sigma_m,\tfrac{1}{2}g]&=\sigma'[\tfrac{1}{2}g]=(\Phi^m\sigma_0]'[\tfrac{1}{2}g]\\&=(2m+1)\sigma_m\qquad m=1,2,\ldots.\end{aligned}\tag{5.5}$$

*Proof.* When $m=1$

$$\begin{aligned}[\sigma_1,\tfrac{1}{2}g]&=g^{-3/2}(u_{xxx}+6uu_x-6xfu_x-6fu+6xf^2)'[\tfrac{1}{2}g]\\&=g^{-1/2}(3u_x-3f)=3\sigma_0\end{aligned}$$

then (5.5) is established for $m=1$.

Suppose (5.5) is established for $m=k-1$, i.e.

$$[\sigma_{k-1},\tfrac{1}{2}g]=\sigma'_{k-1}[\tfrac{1}{2}g]=(\Phi^{k-1}\sigma_0)'[\tfrac{1}{2}g]=(2k-1)\sigma_{k-2}.$$

Notice that $\Phi'[\frac{1}{2}g]=2$ and we have

$$\begin{aligned}[\sigma_k,\tfrac{1}{2}g]&=(\Phi\sigma_{k-1})'[\tfrac{1}{2}g]=\Phi'[\tfrac{1}{2}g]\sigma_{k-1}+\Phi\circ\sigma'_{k-1}[\tfrac{1}{2}g]\\&=2\sigma_{k-1}+\Phi[\sigma_{k-1},\tfrac{1}{2}g]\\&=2\sigma_{k-1}+(2k-1)\sigma_{k-1}\\&=(2k+1)\sigma_{k-1}\end{aligned}$$

which implies (5.5).

*Lemma 6.*

$$[\sigma_m,\Phi^n\tfrac{1}{2}g]=(2m+1)\sigma_{m+n-1}. \tag{5.6}$$

*Proof.* According to lemma 5, this equation is valid for $n=0$. Assume that it is established when $n=k-1$ and let us prove it for $n=k$. In fact, by lemma 4,

$$\begin{aligned}[\sigma_m, \Phi^k \tfrac{1}{2}g] &= \sigma'_m[\Phi^k \tfrac{1}{2}g] - (\Phi^k \tfrac{1}{2}g)'[\sigma_m] \\ &= \sigma'_m[\Phi^k \tfrac{1}{2}g] - \Phi'[\sigma_m]\Phi^{k-1}\tfrac{1}{2}g - \Phi(\Phi^{k-1}\tfrac{1}{2}g)'[\sigma_m] \\ &= \sigma'_m[\Phi^k \tfrac{1}{2}g] - \sigma'_m[\Phi^k \tfrac{1}{2}g] + \Phi\sigma'_m[\Phi^{k-1}\tfrac{1}{2}g] - \Phi(\Phi^{k-1}\tfrac{1}{2}g)'[\sigma_m] \\ &= \Phi[\sigma_m, \Phi^{k-1}\tfrac{1}{2}g] \\ &= (2m+1)\Phi\sigma_{m+k-2} \\ &= (2m+1)\sigma_{m+k-1}.\end{aligned}$$

*Lemma 7.*

$$\begin{aligned}[\Phi^m \tfrac{1}{2}g, \tfrac{1}{2}g] &= (\Phi^m \tfrac{1}{2}g)'[\tfrac{1}{2}g] \\ &= 2m\Phi^{m-1}\tfrac{1}{2}g. \end{aligned} \tag{5.7}$$

*Proof.* Since

$$\Phi[\tfrac{1}{2}g] = 2(u_x - xf) + x(u_x - f)$$
$$(\Phi(\tfrac{1}{2}g))'[\tfrac{1}{2}g] = g$$

which is (5.7) for $m=1$. Assume it is established for $m=k-1$, we prove that it is valid for $m=k$. In fact

$$\begin{aligned}(\Phi^k \tfrac{1}{2}g)'[\tfrac{1}{2}g] &= \Phi'[\tfrac{1}{2}g]\Phi^{k-1}\tfrac{1}{2}g + \Phi(\Phi^{k-1}\tfrac{1}{2}g)'[\tfrac{1}{2}g] \\ &= 2\Phi^{k-1}\tfrac{1}{2}g + 2(k-1)\Phi^{k-1}\tfrac{1}{2}g \\ &= 2k\Phi^{k-1}\tfrac{1}{2}g.\end{aligned}$$

which implies (5.7).

In a similar way to the proof of lemmas 7 and 8 in Li and Zhu (1985) we can prove the following lemma.

*Lemma 8.*

$$[\Phi^m \tfrac{1}{2}g, \Phi^n \tfrac{1}{2}g] = 2(m-n)\Phi^{m+n-1}\tfrac{1}{2}g. \tag{5.8}$$

*Proof of the theorem.*

(i) (5.1) is the direct result of lemma 4.

(ii) According to lemma 6

$$\begin{aligned}[\sigma_m, \tau_n] &= [\sigma_m, 3h\sigma_m + \Phi^n \tfrac{1}{2}g] \qquad \left(h = g^{-1/2}\int g^{3/2}\,\mathrm{d}t\right) \\ &= [\sigma_m, \Phi^n \tfrac{1}{2}g] = (2m+1)\sigma_{m+n-1}.\end{aligned}$$

(iii) According to lemmas 6 and 8

$$\begin{aligned}[\tau_m, \tau_n] &= [3h\sigma_m + \Phi^m \tfrac{1}{2}g, 3h\sigma_n + \Phi^n \tfrac{1}{2}g] \\ &= 3h[\sigma_m, \Phi^n \tfrac{1}{2}g] + 3h[\Phi^m \tfrac{1}{2}g, \sigma_n] + [\Phi^m \tfrac{1}{2}g, \Phi^n \tfrac{1}{2}g] \\ &= 3h(2m+1)\sigma_{m+n-1} - 3h(2n+1)\sigma_{m+n-1} + 2(m-n)\Phi^{m+n-1}\tfrac{1}{2}g \\ &= 2(m-n)(3h\sigma_{m+n-1} + \Phi^{m+n-1}\tfrac{1}{2}g) \\ &= 2(m-n)\tau_{m+n-1}.\end{aligned}$$

This completes the proof.

The above results can be extended to the equation

$$u_t + u_{xxx} + 6uu_x + 6f(t)u - x(f' + 12f^2) - (l' + 12lf) = 0$$

where $f(t)$ and $l(t)$ are arbitrary functions of $t$.

**Acknowledgments**

The author would like to thank Professors Li Yi-shen and G Soliani for many valuable discussions. He would also like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

**References**

Calogero F 1985 Private communication
Chen H H, Lee Y C and Lin J E 1982 *Preprint UMLPF*
Chen H H and Zhu Guo-Cheng 1984 *Plasma preprint UMLPF 85-107*
Fuchssteiner B 1981 *Prog. Theor. Phys.* **65** 861-76
Fuchssteiner B and Oevel W 1982 *J. Math. Phys.* **23** 358-63
Ibragimov N Kh and Shabat A B 1979 *Dokl. Akad. Nauk* **244** 57-61
Li Yi-shen and Zhu Guo-Cheng 1985 *Preprint USTC, Hefei*
Oevel W and Fokas A S 1984 *J. Math. Phys.* **25** 918-22
Olver P J 1980 *Math. Proc. Camb. Phil. Soc.* **88** 71
Stramp W 1984 *J. Phys. Soc. Japan* **53** 4129-37
Tian Chou 1985 *Preprint USTC, Hefei*
Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522-6